Indian Statistical Institute,Bangalore Centre
B.Math.(Hons.)II Year - 2016-17, First Semester
Optimization

Mid Term Exam 14 September 2016, 2 pm - 5 pm.
Instructor: P.S.Datti Max.Marks: 30
NOTE: Answer any **FIVE** questions. WRITE NEATLY.

1. Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 3 & 1 & 5 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 14 \\ 18 \\ 20 \end{pmatrix}$. Obtain the **LU** decomposition of **A** and use it to solve the system $\mathbf{Ax} = \mathbf{b}$. **(3+3)**

2. (i) Let $\mathbf{a}, \mathbf{b}$ be column vectors in $\mathbb{R}^n$ and consider the matrix $\mathbf{A} = \mathbf{I} + \mathbf{ab}^t$. Show that $\mathbf{A}^2 + \alpha\mathbf{A} + \beta\mathbf{I} = \mathbf{O}$ for some real numbers α and β. Hence find $\mathbf{A}^{-1}$, when exists. **(3)**

   (ii) Consider the $n \times n$ matrix

$$\mathbf{B} = \begin{pmatrix} a & b & b & \dots & b \\ b & a & b & \dots & b \\ b & b & a & \dots & b \\ \dots & \dots & \dots & \dots & \dots \\ b & b & b & \dots & a \end{pmatrix},$$

   where a, b are real numbers such that $a \neq b$ and $a + (n-1)b \neq 0$. Solve the system $\mathbf{Bx} = \mathbf{c}$ for a given vector $\mathbf{c} \in \mathbb{R}^n$.(Suggestion: Write a scalar multiple of **B** in the form of the matrix as in (i)) **(3)**

3. (a) Let $\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. Find a 4×3 matrix **Q** satisfying $\mathbf{Q}^t\mathbf{Q} = \mathbf{I}_3$ and an upper triangular 3×3 matrix **R** with all diagonal elements positive such that $\mathbf{A} = \mathbf{QR}$. **(4)**

   (b) Suppose V is an inner product space over $\mathbb{C}$ and **P** is a projection in V. If $(\mathbf{P}x, x) \leq \|x\|^2$ for all vectors $x \in V$, show that **P** is an orthogonal projection. (Suggestion: If $x \in \operatorname{im}\mathbf{P}$ and $y \in \ker \mathbf{P}$, apply the given condition to the vector $x + \lambda y$ for appropriate $\lambda \in \mathbb{C}$.) **(2)**

4. Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2$ be non-zero column vectors in $\mathbb{R}^n$ and define $\mathbf{P} = \mathbf{u}_1\mathbf{u}_2^t + \mathbf{v}_1\mathbf{v}_2^t$. Derive sufficient conditions on the given vectors so that **P** is a projection. Further, derive sufficient conditions on the given vectors so that **P** is an orthogonal projection. **(3+3)**

5. Let $\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \in \mathbb{R}^3$. Let $c = \min \|\mathbf{Ax} - \mathbf{b}\|$, where the minimum is taken over all the column vectors $\mathbf{x}$ in $\mathbb{R}^3$. Determine c and obtain the solution $\mathbf{x}$ with least norm such that $c = \|\mathbf{Ax} - \mathbf{b}\|$. **(2+4)**

6. Let $\mathbf{A} = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}$. Explain in detail why the limit $\lim_{k\to\infty} \mathbf{A}^k$ exists and find the limit. **(3+3)**